FINITENESS CLASSES ARISING FROM GOWERS' FIN_k THEOREM

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Abstract. We study finiteness classes arising from failures of Gowers' FIN_k theorem, extending the work of Brot, Cao, and Fernández-Bretón on Ramsey theoretic finiteness in choiceless set theory [1]. We define the notion of G_k-finiteness – a set X is G_k-finite if there exists a coloring $c : \mathrm{FIN}_k(X) \to 2$ admitting no infinite block family with monochromatic tetris sums. We show that G_1-finiteness coincides with H-finiteness introduced by Brot, Cao, and Fernández-Bretón, and that these classes satisfy G_k-Fin $\subseteq G_{k+1}$-Fin, for all k. Our main result is that these inclusions are not strict. In fact, we prove something stronger: *all G_k-finiteness classes are identical.* Specifically, we prove in ZF that for any set X and any $k \geq 1$, X is G_k-finite if and only if $[X]^{<\omega}$ is Dedekind-finite if and only if X is H-finite. We also show the existence of G-finite sets in some classical Fraenkel-Mostowski permutation models.

1. Introduction

1.1. **Background.** A finiteness class is a class $\mathcal{F}$ of sets satisfying: (i) closure under subsets, (ii) closure under equipotence, (iii) containment of all finite sets, and (iv) $\omega \notin \mathcal{F}$. In the absence of the Axiom of Choice (AC), many non-equivalent notions of finiteness arise, each defining a finiteness class situated between the class Fin and the class D-Fin of all Dedekind-finite sets.

Brot, Cao, and Fernandez-Breton [1] investigated finiteness classes arising from the failure of Ramsey's theorem and Hindman's theorem, introducing the notions of R_n-finiteness and H-finiteness (together with intermediate notions H_2 and H_3). Their paper concludes with several open questions, one of which (Question 5.1(5)) asks:

> "What interesting things can one say about the analogous notions of finiteness arising from Gowers' FIN_k theorem?"

In this paper, we take up this question. We introduce the FIN_k spaces, describe Gowers' theorem and the associated tetris operation, define the corresponding finiteness classes, and investigate where they sit in the hierarchy of choiceless finiteness notions.

1.2. **The FIN_k spaces.** We now introduce the combinatorial spaces FIN_k, first studied in the context of Ramsey theory by Gowers [2] and later developed systematically by Todorcevic [7] and others.

Definition 1.1. *Let X be a set and let $k \geq 1$ be a natural number. The space $\mathrm{FIN})_k(X)$ is defined as:*

$$\mathrm{FIN}_k(X) := \{f : X \to \{0, 1, \dots, k\} | f^{-1}(k) \neq \emptyset \textit{ and } supp(f) \textit{ is finite}\},$$

where $supp(f) = \{x \in X : f(x) \neq 0\}$ denotes the support of f.

In particular, $\mathrm{FIN}_1(X) := \{f : X \to \{0, 1\} | f^{-1}(1) \neq 0 \text{ and } \mathrm{supp}(f) \text{ is finite}\}$, which is naturally identified with the collection $[X]^{<\omega} \setminus \{\emptyset\}$ of non-empty finite subset of X via $f \mapsto \mathrm{supp}(f) = f^{-1}(1)$.

Definition 1.2 (Tetris operation)**.** *For $k \geq 2$, the tetris operation is the map $T : \mathrm{FIN}_k(X) \to \mathrm{FIN}_{k-1}(X)$ defined by*

$$T(f)(x) = \max(f(x) - 1, 0)$$

for each $x \in X$.

To see that T is well-defined, note that if $f \in \mathrm{FIN}_k(X)$ then $f^{-1}(k) \neq \emptyset$, and for any $x \in f^{-1}(k)$ we have $T(f)(x) = k - 1 \geq 1$ so $T(f)^{-1}(k - 1) \supseteq f^{-1}(k) \neq \emptyset$, confirming $T(f) \in \mathrm{FIN}_{k-1}(X)$. Moreover, $\mathrm{supp}(T(f)) = f^{-1}(\{2, 3, 4, \dots, k\}) \subseteq \mathrm{supp}(f)$, so the support remains finite though it may shrink.

Higher iterates of the tetris operation are defined by $T^0 = \mathrm{id}$ and $T^{j+1} = T \circ T^j$. For $f \in \mathrm{FIN}_k(X)$, we have $T^j(f) \in \mathrm{FIN}_{k-j}(X)$ when $1 \leq j \leq k - 1$, and $T^k(f) = 0$ (the zero function, which lies outside all FIN_m spaces).

Definition 1.3 (Block families)**.** *A family $\mathcal{Y} \subseteq \mathrm{FIN}_k(X)$ is called a block family if its elements have pairwise disjoint supports. i.e. $supp(y) \cap supp(y') = \emptyset$ whenever $y, y' \in \mathcal{Y}$ are distinct.*

Remark 1.4. *In the classical setting where $X = \omega$, (equivalently $X = \mathbb{N}$), one typically requires a block sequence $(x_n)_{n<\omega}$ with the additional property that $\max(supp(x_n)) < \min(supp(x_{n+1}))$ for all n. Since we work with arbitrary (potentially non-orderable) sets, we use the weaker notion of block families rather than block sequences.*

Definition 1.5 (Tetris sums)**.** *Let $\mathcal{Y} \subseteq \mathrm{FIN}_k(X)$ be a block family. The tetris sums of $\mathcal{Y}$ denoted $TS(\mathcal{Y})$, consists of all elements of the* *form*

$$\sum_{i=1}^{m} T^{t_i}(y_i),$$

where $m \geq 1$, $y_1, \ldots, y_m \in \mathcal{Y}$ *are distinct,* $t_1, \ldots, t_m \in \{0, 1, \ldots, k-1\}$, *the sum is computed coordinate-wise, and the result likes in* $\mathrm{FIN}_k(X)$ *by demanding at least one* $t_i = 0$.

Since the elements of $\mathcal{Y}$ have pairwise disjoint supports, $\mathrm{supp}(T^{t_i}(y_i)) \subseteq \mathrm{supp}(y_i)$, the supports of the summands are pairwise disjoint, and hence the coordinate-wise sum simply patches together the individual functions. In particular, the sum takes values in $\{0, 1, \ldots, k\}$ so it is well-defined as a function $X \to \{0, \ldots, k\}$ with finite support.

1.3. **Gowers' theorem.** The following theorem was proved by Gowers' [2] in 1992, generalizing Hindman's theorem [4]. Gowers' original motivation was a problem of Erdös.

Theorem 1.6 (Gowers' FIN_k theorem)**.** FIN_k *theorem, [cite]).** Let* $k \geq 1$. *For every coloring* $c : \mathrm{FIN}_k(\omega) \to r$ *with* $r < \omega$ *colors, there exists an infinite block sequence* $(x_n)_{n<\omega}$ *in* $\mathrm{FIN}_k(\omega)$ *such that* $TS(\{x_n : n < \omega\})$ *is monochromatic for* c.

For $k = 1$, Gowers' theorem reduces to Hindman's finite unions theorem (the second statement of Theorem 1.4 in [4]). Indeed $\mathrm{FIN}_1(\omega) \simeq [\omega]^{<\omega} \setminus \{\emptyset\}$, the tetris operation is trivial (T sends all elements to 0), and so the only tetris sums are the finite unions $\mathrm{FU}(\mathcal{Y})$.

For $k \geq 2$, Gowers' theorem is strictly stronger than Hindman's theorem. It implies Hindman's theorem by restriction to FIN_1, but it also captures additional combinatorial structure involving the interplay of different "levels" via the tetris operation. This additional structure is what motivates the study of the finiteness classes in this paper.

We also highlight the following structural properties of the tetris operation. Let $f, g \in \mathrm{FIN}_k(X)$ with disjoint supports.

(i) $T(f + g) = T(f) + T(g)$, where the sums are coordinate-wise (valid due to disjoint supports).
(ii) $\mathrm{supp}(T^j(f)) = f^{-1}(\{j+1, \ldots, k\})$ for $0 \leq j \leq k-1$ and $T^k(f) = 0$.
(iii) The "weight" $w(f) = \sum_{x \in \mathrm{supp}(f)} f(x)$ satisfies $w(T(f)) = w(f) - |\mathrm{supp}(f)|$ when $T(f) \neq 0$.

1.4. **Notation and other basic facts.** We will use $\subset$ instead of $\subseteq$ to emphasize when an inclusion is known to be strict. We also recall the following finiteness classes introduced in [1] which we will make reference to throughout (we have already defined the D-finite sets).

Definition 1.7 (A-finiteness)**.** *We say a set X is A-finite if it is amorphous. That is, X is A-finite if it cannot be partitioned into two disjoint infinite sets. Equivalently, X is A-finite if every subset of X is finite or co-finite. We will regularly refer to such A-finite sets as amorphous rather than A-finite.*

A set which is not A-finite is called A-infinite.

Definition 1.8. *A set X is B-finite if no infinite subset of X is linearly orderable.*

A set which is not B-finite is called B-infinite.

Definition 1.9. *A set X is C-finite if there is no surjection $f : X \twoheadrightarrow \omega$.*

A set which is not C-finite is called C-infinite.

Definition 1.10. *A set X is E-finite if no proper subset of X surjects onto X.*

A set which is not E-finite is called E-infinite.

That all of the above are valid notions of finiteness is standard in the literature [8] [5]. Moreover, we collect the following fact from [1]:

Theorem 1.11. *The following are true:*

(1) *A-Fin $\implies$ B-Fin and A-Fin $\implies$ C-Fin.*
(2) *C-Fin $\implies$ E-Fin.*

Moreover, none of these implications are reversible in ZF.

We also collect some facts about H-finiteness as introduced in [1].

Definition 1.12. *Let X be a set. The collection $FU(X)$ is the set of all finite unions of elements of $[X]^{<\omega}$.*

Definition 1.13. *A set X is H-finite if there exists a coloring $c : [X]^{<\omega} \to 2$ such that for every infinite $Y \subseteq [X]^{<\omega}$ consisting of pairwise disjoint elements, the set $FU(Y)$ is not monochromatic for c.*

2. The G_k finiteness classes

We now define the finiteness classes arising from Gowers' theorem.

Definition 2.1. *Let X be a set and let $k \geq 1$.*

(1) *We say that X is G_k-infinite if for every coloring $c : \mathrm{FIN}_k(X) \to 2$, there exists an infinite block family $\mathcal{Y} \subseteq \mathrm{FIN}_k(X)$ such that $TS(\mathcal{Y})$ is monochromatic for c.*
(2) *We say that X is G_k-finite if it is not G_k-infinite.*

We denote by G_k-Fin the class of all G_k-finite sets.

Proposition 2.2. *For each $k \geq 1$, the class G_k-Fin is a finiteness class.*

Proof. We verify the four conditions.

Closure under subsets. Let X be G_k-finite, and $Y \subseteq X$. Since $\mathrm{FIN}_k(Y) \subseteq \mathrm{FIN}_k(X)$ (any function $Y \to \{0, \dots, k\}$ with finite support can by extended to a function on X with the same finite support by sending the elements of $X \setminus Y$ to 0), a bad coloring $c : \mathrm{FIN}_k(X) \to 2$ restricts to a coloring $c|_{\mathrm{FIN}_k(Y)}$ of $\mathrm{FIN}_k(Y)$. Any infinite block family in $\mathrm{FIN}_k(Y)$ is also an infinite block family in $\mathrm{FIN}_k(X)$ (since the former is the subset of the latter), and so any monochromatic infinite block family in $\mathrm{FIN}_k(Y)$ is one in $\mathrm{FIN}_k(X)$, and so none can exist as otherwise this existence would contradict the G_k-finiteness of X.

Closure under equipotence. Let X be G_k-finite with witness coloring c, and let $\phi : X \to Y$ be a bijection. Then ϕ induces a bijection $\phi_* : \mathrm{FIN}_k(X) \to \mathrm{FIN}_k(Y)$ via $\phi_*(f) = f \circ \phi^{-1}$. The coloring $c' = c \circ \phi_*^{-1} : \mathrm{FIN}_k(Y) \to 2$ witnesses that Y is G_k-finite, since block families and tetris sums are preserved under the bijection.

Finite sets are G_k-finite. If X is finite, then every block family in $\mathrm{FIN}_k(X)$ is finite (since the elements have pairwise disjoint supports all contained in X). Hence, no infinite block family exists in $\mathrm{FIN}_k(X)$, so every coloring vacuously witnesses G_k-finiteness.

ω is not G_k-finite. By Gowers' theorem (Theorem 1.6), ω is G_k-infinite. □

We now show that the first level of the Gowers finiteness hierarchy coincides with the Hindman-type finiteness already studied in [1].

Theorem 2.3. *For any set X, X is G_1-finite if and only if X is H-finite. Consequently, G_1-Fin $= H$-Fin.*

Proof. By definition $\mathrm{FIN}_1(X)$ consists of all functions $f : X \to \{0, 1\}$ with $f^{-1}(1) \neq \emptyset$ and finite support. The map $f \mapsto \mathrm{supp}(f) = f^{-1}(1)$ is a bijection $\mathrm{FIN}_1(X) \to [X]^{<\omega} \setminus \{\emptyset\}$. Indeed, given a non-empty finite subset of X, we can consider the function f in $\mathrm{FIN}_k(X)$ which is 1 on elements of this subset and 0 everywhere else, so this map is surjective. Moreover, two functions $f, g \in \mathrm{FIN}_1(X)$ can only map to the same subset if they have the same values on every $x \in X$. i.e. the map is both injective and surjective. In particular, the map identifies $\mathrm{FIN}_1(X)$ with the non-empty finite subsets of X.

The tetris operation on $\mathrm{FIN}_1(X)$ sends every element to the zero function, therefore, for any block family $\mathcal{Y} \subseteq \mathrm{FIN}_1(X)$, the only valid tetris sums (those remaining in $\mathrm{FIN}_1(X)$) are exactly the finite unions:

$$\mathrm{TS}(\mathcal{Y}) = \mathrm{FU}(\mathcal{Y}),$$

under the earlier identification. Thus, X is G_1-finite iff it is H-finite. $\square$

In addition to this correspondence, we also have the following:

Theorem 2.4. *For each $k \geq 1$, we have G_k-Fin $\subseteq G_{k+1}$-Fin. In other words, G_{k+1}-infinite implies G_k-infinite.*

Proof. Let X be G_k-finite. Let $c : \mathrm{FIN}_k(X) \to 2$ be a coloring witnessing the G_k-finiteness of X. Define $c' : \mathrm{FIN}_{k+1}(X) \to 2$ by $c'(f) = c(T(f))$. This is well-defined since $T : \mathrm{FIN}_{k+1}(X) \to \mathrm{FIN}_k(X)$.

We claim c' witnesses the G_{k+1}-finiteness of X. Suppose for contradiction that there exists an infinite block family $\mathcal{Y} \subseteq \mathrm{FIN}_{k+1}(X)$ such that $\mathrm{TS}(\mathcal{Y})$ is monochromatic for c', say in color i.

Define $\mathcal{Y}' = \{T(y) : y \in \mathcal{Y}\} \subseteq \mathrm{FIN}_k(X)$. We verify that $\mathcal{Y}'$ is an infinite block family in $\mathrm{FIN}_k(X)$:

(i) That each $T(y) \in \mathrm{FIN}_k(X)$ follows from the definition of the tetris operation T.
(ii) Since $\mathrm{supp}(T(y)) \subseteq \mathrm{supp}(y)$ and the elements of $\mathcal{Y}$ have pairwise disjoint supports, the elements of $\mathcal{Y}'$ have pairwise disjoint supports.
(iii) If $y \neq y'$ in $\mathcal{Y}$, then $T(y) \neq T(y')$ since they have disjoint supports. Thus the map $y \mapsto T(y)$ on $\mathcal{Y}$ is injective, and since $\mathcal{Y}$ is infinite, so too is $\mathcal{Y}'$.

We will now show that every tetris sum of $\mathcal{Y}'$ in $\mathrm{FIN}_k(X)$ has color i under c, contradicting the assumption that c witnesses G_k-finiteness.

Let $s = \sum_{j=1}^{m} T^{a_j}(T(y_j))$ be a tetris sum from $\mathcal{Y}'$ lying in $\mathrm{FIN}_k(X)$ where $y_1, \ldots, y_m \in \mathcal{Y}$ are distinct and $a_j \in \{0, \ldots, k-1\}$. Since the elements have pairwise disjoint supports

$$s = \sum_{j=1}^{m} T^{a_j+1}(y_j).$$

Now define $s' = \sum_{j=1}^{m} T^{a_j}(y_j)$. We claim $s' \in \mathrm{FIN}_{k+1}(X)$ and that $T(s') = s$.

For the first claim: each $T^{a_j}(y_j)$ has values in $\{0, \ldots, k+1-a_j\}$ and since the supports are pairwise disjoint, s' takes values in $\{0, \ldots, k+1\}$. Moreover, $s \in \mathrm{FIN}_k(X)$ means $s^{-1}(k) \neq \emptyset$, so for some j_0 and some $x_0 \in \mathrm{supp}(y_{j_0})$ we have $T^{a_{j_0}+1}(y_{j_0})(x_0) = k$, which means $y_{j_0}(x_0) =$

$k+a_{j_0}+1$. Since $y_{j_0} \in \mathrm{FIN}_{k+1}(X)$, we need $k+a_{j_0}+1 \leq k+1$, giving $a_{j_0}=0$ so $T^{a_{j_0}}(y_{j_0}) = y_{j_0}$, and $s'(x_0) = y_{j_0}(x_0) = k+1$ (by disjointness of supports). Thus, $s' \in \mathrm{FIN}_{k+1}(X)$.

For the second claim: since the elements have disjoint supports and T acts coordinate-wise,

$$T(s') = T\left(\sum_{j=1}^{m} T^{a_j}(y_j)\right) = \sum_{j=1}^{m} T^{a_j+1}(y_j) = s.$$

Therefore s' is a tetris sum from $\mathcal{Y}$ in $\mathrm{FIN}_{k+1}(X)$ and by our monochromatic assumption, $c'(s') = i$. By definition then, $c'(s') = c(T(s')) = c(s) = i$. So, in particular, every tetris sum of $\mathcal{Y}'$ in $\mathrm{FIN}_k(X)$ has $c(s) = i$, contradiction our assumption that c witnessed the G_k-finiteness of X. ☐

The basic results can then be summarized as:

Corollary 2.5. *The following chain of inclusions holds provably in* ZF*:*

$$\text{Fin} \subseteq H\text{-Fin} = G_1\text{-Fin} \subseteq G_2\text{-Fin} \subseteq \ldots D\text{-Fin}.$$

Proof. Every finite set is G_1-finite by definition. The equality G_1-Fin $=$ H-Fin is Theorem 2.3. That the chain is ascending is Theorem 2.4. It remains to show G_k-Fin $\subseteq$ D-Fin for each k. But a standard proof shows that D-Fin is the largest finiteness class [3]. We reproduce the proof briefly here so as to be self-contained:

Let I-Fin be a finiteness class of sets and let $X \in I$-Fin. Suppose for a contradiction that X is not Dedekind-finite. Then, there exists an injection $i : \omega \hookrightarrow X$, but since I-Fin is a finiteness class, and hence closed under subsets, $\omega \in I$-Fin. Thus, I-Fin cannot be a finiteness class. ☐

3. Relations to other finiteness classes

In the previous section we established that the G_k-finiteness hierarchy was ascending (Theorem 2.4) and that G_1-finiteness was the same as the H-finiteness from [1]. We will now spend some time establishing the relations between the G_k-finiteness classes and the other finiteness classes introduced in Section 1.4.

Corollary 3.1. *For each $k \geq 1$, every C-finite set is G_k-finite. i.e.* C-Fin $\subseteq G_k$-Fin.

Proof. From [1], every C-finite set is H-finite. Hence by Corollary 2.5, we have the result. ☐

More interestingly:

Proposition 3.2. *Let X be a set and $k \geq 1$. If $[X]^{<\omega}$ is D-finite, then $\mathrm{FIN}_k(X)$ is D-finite.*

Proof. We prove the contrapositive. Suppose $\mathrm{FIN}_k(X)$ is D-infinite. Then there is a countable sequence of functions $(f_n)_{n<\omega}$ in $\mathrm{FIN}_k(X)$. We will construct a countable sequence of finite subsets of X from this sequence as follows. Let $Y_1 = \mathrm{supp}(f_1)$. Since $f_1 \in \mathrm{FIN}_k(X)$, Y_1 is finite. Hence $Y_1 = \mathrm{supp}(f_1)$ can only support finitely many functions in $\mathrm{FIN}_k(X)$. Let m be the number of functions which can be supported on Y_1. If we consider the first $m+1$ functions in the sequence $(f_n)_{n<\omega}$, by the pigeonhole principle, there exists a first function f_{n_2} not supported on Y_1. Let $Y_2 = \mathrm{supp}(f_{n_2})$. Clearly $Y_2 \neq Y_1$. Now, both Y_1, Y_2 are finite, so $Y_1 \cup Y_2$ is finite. By the same logic from earlier there will eventually be a function f_{n_3} with support strictly outside $Y_1 \cup Y_2$. Let $Y_3 = \mathrm{supp}(f_{n_3})$. Iterating in this way, we construct a non-trivial countably infinite sequence of finite subsets of X. This sequence $(Y_n)_{n<\omega}$ is a witness to the failure of the D-finiteness of $[X]^{<\omega}$. In particular if $\mathrm{FIN}_k(X)$ is D-infinite, $[X]^{<\omega}$ is D-infinite. Equivalently $[X]^{<\omega}$ D-finite implies $\mathrm{FIN}_k(X)$ D-finite. ☐

This leads us to our main theorem for this section:

Theorem 3.3 (Collapse of the G_k-finiteness hierarchy)**.** *For any set X and $k \geq 1$, the following are equivalent:*

(1) X is G_k-finite.
(2) $[X]^{<\omega}$ is D-finite.
(3) X is G_1-finite (equivalently H-finite).

In particular, G_k-Fin $= G_1$-Fin $= H$-Fin.

Before proving this theorem it is helpful to introduce some definitions and basic facts.

Definition 3.4. *For a set X and $k \geq 1$, the top-level coloring $c_k : \mathrm{FIN}_k(X) \to 2$ is defined by $c_k(f) = \lfloor \log_2 |f^{-1}(k)| \rfloor \bmod 2$. This is well-defined since any function f in $\mathrm{FIN}_k(X)$ must achieve the value k on at least one element of its support. i.e. $|f^{-1}(k)| \geq 1$, so the log is defined.*

The following fact holds from the definition and properties of tetris sums:

Lemma 3.5. *Let $\mathcal{Y} \subseteq \mathrm{FIN}_k(X)$ be a block family and let $s = \sum_{i=1}^{m} T^{t_i}(y_i)$ be a tetris sum from $\mathcal{Y}$ with y_i distinct and $t_i \in \{0, \dots, k-1\}$. Then*

$$s^{-1}(k) = \sqcup_{i=1; t_i=0}^{m} y_i^{-1}(k).$$

Consequently

$$|s^{-1}(k)| = \sum_{i=1;t_i=0}^{m} |y_i^{-1}(k)|.$$

We also collect the seemingly unrelated (and somewhat trivially appearing lemma):

Lemma 3.6. *Let a, b be positive integers with $a = b$. Then*

$$\lfloor \log_2(a+b) \rfloor = 1 + \lfloor \log_2 a \rfloor.$$

In particular $\lfloor \log_2(a+b) \rfloor \neq \lfloor \log_2 a \rfloor$.

Proof. Since $a = b$, $a + b = 2a$. So

$$\log_2(a+b) = \log_2(2a) = \log_2(2) + \log_2(a) = 1 + \log_2(a).$$

But adding 1 doesn't change the floor, so in particular the result follows by taking the floor of both sides of this equality. $\square$

We are now ready for the proof of the main theorem.

Proof of Theorem 3.3. (2) $\iff$ (3) is the statement of part of Theorem 3.2 from [1].

(1) $\implies$ (3) comes from Corollary 2.5.

Hence, it suffices to establish (2) $\implies$ (1) to complete the proof. Assume that $[X]^{<\omega}$ is D-finite. We show that the top-level coloring c_k (Definition 3.4) witnesses that X is G_k-finite. Thus, suppose toward a contradiction that there exists an infinite block family $\mathcal{Y} \subseteq \mathrm{FIN}_k(X)$ such that $\mathrm{TS}(\mathcal{Y})$ is monochromatic for c_k, for a color $i \in 2$.

Step 1: All elements of $\mathcal{Y}$ have distinct top-level sizes. Take any two elements $y, y' \in \mathcal{Y}$. Since y, y' necessarily have disjoint supports and the definition of tetris sums requires at least one summand that is not "tetrized," $y + y'$ is a valid tetris sum from $\mathcal{Y}$. By Lemma 3.5, $|(y+y')^{-1}(k)| = |y^{-1}(k)| + |y'^{-1}(k)|$. Since y itself is a valid tetris sum, monochromaticity gives $c_k(y) = c_k(y + y') = i$. i.e.

$$\lfloor \log_2 |y^{-1}(k)| \rfloor = \lfloor \log_2(|y^{-1}(k)| + |y'^{-1}(k)|) \rfloor \bmod 2.$$

But now, if $|y^{-1}(k)| = |y'^{-1}(k)|$, by Lemma 3.6,

$$\lfloor \log_2 |y^{-1}(k)| \rfloor = 1 + \lfloor \log_2 |y^{-1}(k)| \rfloor \bmod 2,$$

which is absurd. Thus, $|y^{-1}(k)| \neq |y'^{-1}(k)|$ for all distinct $y, y' \in \mathcal{Y}$.

Step 2: There is a injection into ω. In particular then, the function $\phi : \mathcal{Y} \to \omega$ defined by sending $y \in \mathcal{Y}$ to its level set at k is an injection. But each level set at k is an element of $[X]^{<\omega}$, so $[X]^{<\omega}$ cannot be D-finite as we assumed, a contradiction. $\square$

In particular, the analysis in the subsequent section does not rely on any special properties of k.

4. Analysis in Fraenkel-Mostowski models

We finally investigate the G_k-finiteness classes in several classical Fraenkel-Mostowski permutation models. Throughout this section, we work in $\mathsf{ZFA} + \mathsf{AC}$ with a countable set A of atoms and define our models via the class HS of hereditarily symmetric sets with respect to a given group of permutations of A and finite supports.

By the Jech-Sochor transfer theorem the relevant statements about finiteness classes transfer from these models to models of ZF; the details of this transfer are exactly as described in [6].

4.1. **The first Fraenkel model: strongly amorphous atoms.** Recall that the first Fraenkel model $\mathcal{N}1$ is the model hereditarily symmetric (HS) with respect to the full permutation group $G_1 = S_A$ and finite supports. The following result we will make use of is due to Truss [8].

Lemma 4.1. *The set A of atoms is strongly amorphous in $\mathcal{N}1$.*

Proof. We will first show that A is amorphous. Let $X \subseteq A$ be HS. Let $E \subset A$ be a finite support for X. Let a, b be any two atoms in $A \setminus E$. Consider the transposition $\pi = (a\ b)$. Because π only swaps a and b it fixes every element of E pointwise. Since E supports X, any permutation fixing E pointwise must preserve X. Thus $\pi X = X$. By the definition of the group action on sets, $a \in X \iff \pi(a) \in \pi(X)$. Since $\pi(a) = b$, and $\pi(X) = X$ this condition is akin to demanding $a \in X \iff b \in X$. In particular, the property of belonging to X is uniform across all elements outside the support of E. That is, every element of $A \setminus E$ is in X, or no elements of $A \setminus E$ are in X. Since E is finite, X is finite, or co-finite. Thus, every subset of A is finite or co-finite, proving A is amorphous.

To prove A is strongly amorphous, we must show that any partition of X into infinitely many finite sets is such that all but finitely many pieces in the partition are singletons.

Let Π be a partition of A into finite sets and suppose Π is HS. Let E be a finite support for Π. Let $P \in \Pi$ be a piece of the partition disjoint from E. Such a P must exist since E is finite. Suppose for the sake of contradiction that P is not a singleton and let $a, b \in P$. Because P is finite, we can choose $c \in A \setminus E$ such that $c \notin P$. Consider the transposition $\pi = (b\ c)$. Clearly π fixes E pointwise, so $\pi(\Pi) = \Pi$ since E is a support for Π. But this implies that $P' = \pi(P) = (P \setminus \{b\}) \cup \{c\}$ is an element of the partition. But this is impossible since P, P' share the element a. Thus, our assumption that $|P| > 1$ is false, and so only finitely many non-singleton elements (in particular those subsets of the support for the partition) exist. □

Using the above, we can establish the following:

Theorem 4.2. *In the first Fraenkel model ($\mathcal{N}1$), the set of atoms is G_k-finite for every $k \geq 1$.*

Proof. Step 1: Structure of the block families in $\mathrm{FIN}_k(A)$*.* We first observe that any infinite block family $\mathcal{Y} \subseteq \mathrm{FIN}_k(A)$ in the model is highly constrained. The supports of elements of $\mathcal{Y}$ form an infinite family of pairwise disjoint finite subsets of A, and their union is a co-finite subset of A. Since A is strongly amorphous, any partition of a co-finite subset of A into finite pieces has all but finitely many pieces being singletons.

Therefore, all but finitely many elements of $\mathcal{Y}$ have singleton support. An element of $\mathrm{FIN}_k(A)$ with singleton support $\{a\}$ is uniquely determined by the value it assigns to a which must be k (since elements of $\mathrm{FIN}_k(A)$ are required to attain value k.) Thus we denote these functions as δ_a^k (the function sending a to k and everything else to 0). Moreover, all but finitely many elements of $\mathcal{Y}$ have the form δ_a^k for some $a \in A$.

Step 2: Constructing a bad coloring. Define $c : \mathrm{FIN}_k(A) \to 2$ by $c(f) = |f^{-1}(1)| \bmod 2$. We claim this coloring is HS. To see this, we observe that any permutation $\pi \in S_A$ preserves the cardinality of level sets of a finite function. i.e. $c(\pi(f)) = c(f)$ for all $f \in \mathrm{FIN}_k(A)$. Thus, $\emptyset$ is a support for c and so $c \in \mathrm{HS}$.

Step 3: No infinite block family has monochromatic tetris sums. Now, let $\mathcal{Y} \subseteq \mathrm{FIN}_k(A)$ be an infinite block family in this model. By the first step, we delete those non-singleton elements and may assume without loss of generality that $\mathcal{Y} = \{\delta_{a_n}^k : m < \omega\}$, where this well-ordering lives in the ground model. We consider the following tetris sums:

- $s_1 = T^0(\delta_{a_1}^k) = \delta_{a_1}^k$. Then $s_1^{-1}(1) = \emptyset$, so $c(s_1) = 0$.
- $s_2 = T^0(\delta_{a_1}^k) + T^{k-1}(\delta_{a_2}^k)$. Now $T^{k-1}(\delta_{a_2}^k)(a_2) = \max(k - (k - 1), 0) = 1$, and all other values are 0 so $s_2(a_1) = k$ and $s_2(a_2) = 1$ giving $s_2^{-1}(1) = \{a_2\}$ and so $c(s_2) = 1$.

Since $c(s_1) \neq c(s_2)$, the set $\mathrm{TS}(\mathcal{Y})$ is not monochromatic for c. As $\mathcal{Y}$ was arbitrary, c witnesses that A is G_k-finite.

□

Corollary 4.3. *In* ZF *every strongly amorphous set is G_k-finite, since we only relied on the strong amorphousness of A to prove Theorem 4.2.*

4.2. **The second Fraenkel model: Russell sets.** Recall that the second Fraenkel model $\mathcal{N}2$ is constructed by partitioning $A = \bigcup_{n<\omega} P_n$ where each $P_n = \{a_n, b_n\}$ is a pair, and taking G_2 to be the group of permutations which set-wise fix each P_n. The model is then the sets HS with respect to the finite supports.

Again, we characterize the atoms in this model before presenting our main result for this part. This result is from the standard literature, e.g. [5]

Lemma 4.4. *The set of atoms in $\mathcal{N}2$ is a Russell set. i.e. The set $\{P_n : n < \omega\}$ has no choice function in $\mathcal{N}2$.*

Proof. To see $\{P_n : n < \omega\}$ is HS, it suffices to observe that any permutation $\pi \in G_2$ fixes the pairs, and so $\emptyset$ supports $\{P_n : N < \omega\}$. Suppose now that $C \subset A$ is a family containing precisely one element from each pair. Suppose C is HS and let E be a finite support. Since E is finite, there is a pair P_n which does not intersect E. Consider the transposition $\pi = (a_n\ b_n)$. This is a legitimate transposition fixing the pairs set-wise, and since it fixes E, $\pi(C) = C$. But then a_n, b_n are both elements of C contradicting that C selects precisely a single element from each pair. □

Using this fact we have:

Theorem 4.5. *In the second Fraenkel model, the set A of atoms is G_k-infinite for every $k \geq 1$.*

Proof. Step 1: Structure of the symmetric elements. Let $c : \mathrm{FIN}_k(A) \to 2$ be a coloring in $\mathcal{N}2$ and let $F \subset A$ be a finite support. Let N be large enough that $F \subseteq \bigcup_{n \leq N} P_n$. For $n \geq N$, any permutation $\pi \in G_2$ that fixes F pointwise is free to swap the elements of P_n. Consequently, for any $f \in \mathrm{FIN}_k(A)$ whose support is contained in some P_n with $n \geq N$, the transposition τ_n of P_n fixes F and hence fixes c, so $c(f) = c(\tau_n(f))$. This means that for $n \geq N$, two elements of $\mathrm{FIN}_k(A)$ supported on P_n that differ only by swapping $a_n \leftrightarrow b_n$ receive the same color.

Step 2: Constructing a monochromatic block family. Define for each $n \geq N$, the element $y_n \in \mathrm{FIN}_k(A)$ by

$$y_n(a_n) = k, \quad y_n(b_n) = k, \quad y_n(x) = 0 \text{ for } x \notin P_n.$$

Each y_n has support P_n, so $\{y_n : n \geq N\}$ is an infinite block family. Observe that every tetris sum from $\{y_n : n \geq N\}$ is a function that on each pair P_n involved is constant (both atoms of P_n receive the same value namely $k - t_n$ for the chosen tetris iteration t_n). Indeed:

$$s = \sum_{j=1}^{m} T^{t_j}(y_{n_j}),$$

where $T^{t_j}(y_{n_j})$ sends both a_{n_j}, b_{n_j} to $k-t_j$. We define the type of such a tetris sum s to be the multiset of non-zero values $\{k-t_1, \dots k-k-t_m\}$ (each appearing with multiplicity 2 since each pair contains two atoms at the same level). Any permutation $\sigma \in G_2$ which fixes F permutes atoms within each pair, which preserves the type. Therefore, the color $c(s)$ depends only on the ordered tuple of the values $(k-t_1, \dots, k-t_m)$.

Since the pair structure ensures that every relevant function is constant on pairs, the color $c(s)$ depends only on this multiset of values assigned to the pairs involved. We now apply Gowers' $\mathrm{FIN}_k(\omega)$ theorem for ω (provable in ZF): define a coloring $d : \mathrm{FIN}_k(\omega) \to 2$ by $d(g) = c(\hat{g})$ where for $g \in \mathrm{FIN}_k(\omega)$, $\hat{g} \in \mathrm{FIN}_k(A)$ is the function which sends a_{N+m}, b_{N+m} to the image of m under g.

We observe that $T(\hat{g}) = \widehat{T(g)}$. Moreover for a block sequence $(g_n)_{n\in\omega}$ in $\mathrm{FIN}_k(\omega)$, the family $\hat{g}_n$ is a block family in $\mathrm{FIN}_k(A)$ (since the sequence is such that it is mapped to disjoint pairs). But now, d is defined on $\mathrm{FIN}_k(\omega)$ and admits a monochromatic infinite block sequence, and so lifting this via the hat operation gives a monochromatic block family in $\mathrm{FIN}_k(A)$ for c as desired.

□

Remark 4.6. *The key ingredient in the above proof is that that Russell-set structure provides a canonical way to double each coordinate, and thus lift Gowers' theorem on ω.*

4.3. **The ω-Fraenkel model.** Recall that the ω-Fraenkel model partitions $A = \bigcup_{m<\omega} A_m$ into countably many infinite sets, with permutation group G_3 of permutations which fix each A_m set-wise. This time we do not need to establish a standard fact about the atoms. Instead, we state the main theorem:

Theorem 4.7. *In the ω-Fraenkel model, the set A of atoms is G_k-finite for all $k \geq 1$.*

Proof. Step 1: Define the bad coloring. Define $c : \mathrm{FIN}_k(A) \to 2$ by $c(f) = |f^{-1}(1)| \bmod 2$. Since every permutation is a bijection, the cardinality of level sets are preserved. i.e. $c(\pi(f)) = c(f)$ for all $\pi \in G_3$ so the empty set is a support for c.

Step 2: The structure of an infinite block family. Let $\mathcal{Y} \subseteq \mathrm{FIN}_k(A)$ be an arbitrary infinite block family in the model, with finite support

$F \subset A$. Since F is finite and the elements of $\mathcal{Y}$ have pairwise disjoint finite supports, all but finitely many elements $y \in \mathcal{Y}$ must have a support strictly disjoint from F. We restrict our attention to these elements.

We claim that for every such y, its support must be a singleton.

1. First, we note that $\text{supp}(y)$ must be confined to a single A_m. Suppose for a contradiction, this is not the case. Then there exist m, m' such that there exists $a \in \text{supp}(y) \cap A_m$ and $c \in \text{supp}(y) \cap A_{m'}$. Because the block A_m is infinite, and both $F, \text{supp}(y)$ are finite, we can choose an atom $b \in A_m$ such that $b \notin F \cup \text{supp}(y)$. Consider the transposition $\pi = (a\ b)$. This permutation only swaps elements in A_m so it is an element of the permutation group G_3. Moreover, π fixes F pointwise, so $\pi(\mathcal{Y}) = \mathcal{Y}$. That is, the permuted function $\pi(y)$ is also an element of $\mathcal{Y}$. However, π fixes $c \in A_{m'}$ and so $y(c) = \pi(y)(c)$ so the supports of $y, \pi(y)$ overlap. Since they are elements of a block family, they must be the same function. i.e. $\pi(y) = y$. But this is a contradiction as $y(a) \neq 0$ but $\pi(y)(a) = y(b) = 0$ since b was chosen outside the support of y. Thus, the support of y must be contained in a single A_m.
2. Second, we claim $\text{supp}(y)$ is exactly a singleton. Towards a contradiction, let $\text{supp}(y)$ contain two distinct atoms $a, c \in A_m$. We again choose $b \in A_m \setminus (F \cup \text{supp}(y))$ and consider the transposition $\pi = (a\ b)$. Again, $\pi \in G_3$ and π fixes c, so again, $\pi(y) = y$, which gives the same contradiction as before.

Now, because $y \in \text{FIN}_k(A)$ it must attain the value k, and since its support is singleton, we know that for all but finitely many elements of the block family $\mathcal{Y}$ (except those which have supports inside the support of $\mathcal{Y}$ itself), have the form δ_a^k from before where $a \notin F$.

Step 3: Breaking monochomaticity. Having established that our infinite block family contains an infinite sequence of singletons $\{\delta_{a_n}^k\}$ we invoke the argument from Section 4.1 to obtain the desired result. □

5. Summary of results and open questions

Our results can be summarized by the following figure:

Although the G_k-finiteness hierarchy collapses, this leaves us with several questions.

Question 5.1. *Is it possible to define non-collapsing finiteness classes from Gowers' theorem by using more refined colorings? For instance, one might consider colorings of* $\text{FIN}_k(X)$ *with more than 2 colors?*

In our proofs, we exploited arithmetic modulo 2 when dealing with 2-colorings, so it is perhaps reasonable to expect that these proofs can be modified for arbitrary n-colorings by simply exploiting arithmetic modulo n. If this is the case, then it is reasonable to suspect that the answer to the above is negative. However, we can ask about modifying G_k finiteness to be non-collapsing in the following sense:

Question 5.2. *What hapapens if one modifies the definition of G_k-finiteness by requiring monochromaticity of only restricted tetris sums? For instance, at most n-summand tetris sums?*

Alternatively, in the case of a negative answer to the previous question:

Question 5.3. *The proof of the collapse theorem (Theorem 3.3) shows that the top-level set is the decisive invariant. Are there natural "non-stratified" Ramsey-theoretic invariants on $\mathrm{FIN}_k(X)$ which yield distinct finiteness classes for distinct k? What properties must such invariants possess?*

Acknowledgments

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